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Fixed Point Results for Non-Commuting Mappings in Metric Space via W-Distance

Abstract

In this paper, some unique common fixed point results are provedfor non-commuting JSR and JSR^*mappings in the complete metric space via$w$-distance. In support of the results some examples are also given.

Keywords: W-Distance, Weakly Commuting, S-JSR(P) Mappings, Fixed Point.

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Introduction

The well-known Banach contraction principle, which declares thaton a complete metric space foreach single-valued contraction self-mapping, always there exists a unique fixed point. This basic principle has been exploited and generalized by many researchers using different contraction conditions, applying different mappings in different spaces.

Review of Literature

Nadler [8] has used the concept of Hausdorff metric and obtained a multi-valued version of the Banach contraction principle .Among others Husain and Latif [2], Feng and Liu [1] have generalized Nadler's fixed point result without using the Hausdorff metric.On the other hand, Kannan [4] has proved an interesting fixed point result for single-valued maps in the setting of metric spaces which is not an extension of the Banach contraction principle .While Latif and Beg [6] have obtained a multivalued version of Kannan's fixed point result. In 1996 the team of Kada, Suzuki and Takahashi [3] came with a new and more generalized concept of w-distance hence, many earlier results are improved. Simultaneously Suzuki and Takahashi [12] worked on weakly contractive maps for single and multi-valued functions and produced some important generalizations of Banach contraction principle and based Nadler's results. Parallel to this work there co-researchers Suzuki [13] improve the Kannan's fixed point results by using w-distance. After that a bulk of investigations have been observed [5] [10][13] and [15].

Applying the concept of w-distence Abdul Latif et al. [7]proved some fixed point and common fixed point results for multi-valued maps with the setting of metric spaces, by which they generalized and improve many results including the results of Latif and Beg [6], Suzuki [13], Kannan [4].

Till then no work is reported in this field.

Aim of the Study

Our aimis to consider non-commuting JSR and JSR^*mappings with w-distance in complete metric space and proved unique common fixed point results.We have furnish some examples in support of our main results.

2. Preliminaries

A bulk of literature exist with commuting and non-commuting mappings.We are defining non-commuting pair of maps as JSR and JSR^* maps which is more improved than the known mappings.

On a metric space the concept of w-distance was introduced by Kada et al.[3] in the following manner:

Let $p: X \times X \to [0, \infty)$ be a function over a metric space (X, d), then$p$ is called w- distance if

1. $\forall\ x, y, z \in \mathrm{X},\ p(x, z) \le p(x, y) + p(y, z)$
2. $\forall\ x \in X$ and $y_n \to y$ in X,$p(x, y) \le \liminf p(x, y_n)$, that is p is lower semi continuous with respect to the second variable y

3. for any given $\varepsilon > 0$, there must be a $\delta > 0$ such that $p(x,z) \le \delta$ and $p(z,y) \le \delta \Longrightarrow p(x,y) \le \varepsilon$

Clearly, everymetric is a w-distance but not conversely.

Definition 2.1

A pair (S,T) of self-mappings S and T ona metric space$(X,d)$is said to beweakly commuting if and only if

$d(STx\,,TSx) \le d(Sx,Tx)$for each x in X.

Definition 2.2

Let S and Tbe theself-mappings ona metric space (X,d) with a w-distance p,then$S$ and$T$ are said to be p-weakly commuting if and only if

$max\,[p(STx,TSx), p(TSx\,,STx)\,] \le p(Sx\,,Tx)$ for each x in X.

Definition 2.3

Let S and Tbe theself-mappings ona metric space $(\mathrm{X,d})$ .Then S and T are said to be weakly compatible if and only if each sequence $\{x_n\}$ such that for some$t$ in X.

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \mathrm{t} \implies \lim_{n\to\infty} d(STx_n\;\; TSx_n) = 0$$

Definition 2.4

Let S and T be the self-mappings ona metric space $(X,d)$with w-distance p,then S and T are said to be (p) compatible if every sequence $\{x_n\}$ such that for some t in X

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ as } n \to \infty$$

$$\Longrightarrow max\,[p(STx_n, TSx_n), p(TSx_n\;\,, STx_n)] \ge 0, \text{as } n \to \infty$$

Definition 2.5

The pair(S,T) of two self-mappings S and Ton a metric space(X,d) is said to be S-JSR mappings if and only if each sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \mathrm{t} \text{ for some } t \text{ in } X$$

$$\Longrightarrow \alpha d(STx_n\;\,,Tx_n\,) \le \alpha d(SSx_n, Sx_n)$$

where $\alpha = \lim\sup$ or $\lim\inf$.

Definition 2.6

The pair(S,T) of two self-mappings S and T on a metric space (X,d) is said to be S-$JSR_{(p)}$ mappings if and only if each sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \text{ in } X$$

$$\Longrightarrow \max\{\alpha p(STx_n, Tx_n), \alpha p\,(Tx_n, STx_n)\}$$

$$\le \max\{\alpha p(Sx_n, Sx_n), \alpha p(Sx_n\;,SSx_n)\}$$

where $\alpha = \lim\sup$ or $\lim\inf$

Definition 2.7

The pair(S,T) of two self-mappings S and T on a metric space (X,d) is said to be S-$JSR^{*}_{(p)}$ mappings if and only if each sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \text{ in } X$$

$$\Longrightarrow \max\{\alpha p(TSx_n, STx_n), \alpha p\,(STx_n, TSx_n)\}$$

$$\le \max\{\alpha p(SSx_n, TTx_n), \alpha p(TTx_n\;,SSx_n)\}$$

where $\alpha = \lim\sup$ or $\lim\inf$

Now we give some lemma which are useful in our main results.

Lemma 2.1 (see [3] and [13])

If (X,d)be a metric space, p be a w-distance on X, $\{x_n\}$, $\{y_n\} \subset X$ be sequences and $\{\alpha_n\},\{\beta_n\} \subset (0,\infty)$be sequences such that$\alpha_n \to 0$ and $\beta_n \to 0$ and for $x,y,z \in X$. Then we have the following conditions:

1. $p(x_n,y) \le \alpha_n, p(x_n,z) \le \beta_n, \forall\, n \in N \Longrightarrow y = z$.Particularly, if$p(x,y) = 0, p(x,z) = 0 \implies y = z$.
2. $p(x_n,y_n) \le \alpha_n, p(x_n\;,z) \le \beta_n \forall\, n \in N \Longrightarrow y_n \to z$.
3. $p(x_n,x_m) \le \alpha_n, \forall\, n,m \in N$ with $m > n \Longrightarrow \{x_n\}$ is a Cauchy sequence.
4. $p(y,x_n) \le \alpha_n, \forall n \in N \Longrightarrow \{x_n\}$is a Cauchy sequence.

Lemma 2.2

If (X,d)be a metric space, p be a w-distance on $X$and let S and T be self mappings on X, satisfying $Tx_n = Sx_{n+1}$ for $n = 0,1,2,\ldots$,assume that there exists a continuous self mapping ξ of $[0,\infty]$ such that

$p(Tx,Ty) \le \xi\big(p(Sx,Sy)\big)$

(2.2.1)

for all $x,y \in X$ and for each $t > 0$

$\xi(t) < t$

(2.2.2)

Then

(A) for an arbitrary $\epsilon > 0$, there exist positive integer m,s such that $m \le n < s$ implies $p(Tx_n\;,Tx_s) < \epsilon$.

(B) the sequence $\{Tx_n\}$ is a Cauchy sequence.

Proof

We have

$$p(Tx_n, Tx_{n+1}) \le \xi\big(p(Sx_n, Sx_{n+1})\big)$$
$$= \xi\big(p(Tx_{n-1}, Tx_n)\big)$$
$$< p(Tx_{n-1}, Tx_n)$$

for$n = 1,2,3,\ldots$.. Thus $\{p(Tx_n, Tx_{n+1})\}$ is a decreasing sequence of non negative real number and there exists non negative real number λ such that

$$\lim_{n\to\infty}(n+1)p(Tx_n, Tx_{n+1}) = \lambda,$$

Let $\lambda > 0$, then the inequality

$$p(Tx_n, Tx_{n+1}) \le \xi\big(p(Tx_{n-1}, Tx_n)\big)$$

Now the continuity of ξ we have$\lambda < \xi(\lambda) < \lambda$,which is contradiction.

Therefore $\lambda = 0$ so $p(Tx_n, Tx_{n+1}) \to 0$ as $n \to \infty$.

(A) Now suppose that (A) does not hold. Then, there exists an $\epsilon > 0$ such that for all sufficiently large positive integer$k$, there exist positive integers $s_k, n_k$with $k \le n_k < s_k$such that

$\epsilon \le \big(p(Tx_{nk}, Tx_{sk})\big), \big(p(Tx_{nk}, Tx_{nk-1})\big) < \epsilon$

(2.2.3)

From (2.2.3), we have

$$p(Tx_{nk}, Tx_{sk}) \to \epsilon \text{ and } p(Tx_{nk}, Tx_{nk-1}) \to 0 \text{ as } k \to \infty$$

And$p(Tx_{nk}, Tx_{sk}) \le p(Tx_{nk}, Tx_{nk+1}) + p(Tx_{nk+1}, Tx_{sk})$

$$\le p(Tx_{nk}, Tx_{nk+1}) + \xi\big(p(Tx_{nk+1}, Tx_{sk})\big)$$
$$\le p(Tx_{nk}, Tx_{nk+1}) + \xi\big(p(Tx_{nk}, Tx_{sk-1})\big) \quad (2.2.4)$$

By the hypothesis and (2.2.4),we obtain $\epsilon \le \xi(\epsilon) < \epsilon$. This is contradiction therefore (A) holds.

Alsowe have from the third condition of the definition of a w-distance $p$and(A) that $\{Tx_n\}$ is a Cauchy sequence.

Lemma 2.3

If (X,d)be a metric space, p be a w-distance on X, let S and T be self-mappings on Xsuch that$Tx_n = Sx_{n+1}$ for $\mathrm{n} = 0,1,2,\ldots\ldots$, with the following conditions: for given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \le p(Sx,Sy) < \epsilon + \delta \Rightarrow p(Tx,Ty) < \epsilon, \quad (2.3.1)$$

$$p(Sx,Sy) < \epsilon \Rightarrow p(Tx,Ty) \le 1/2\; p(Sx,Sy) \quad (2.3.2)$$

The

(C) For an arbitrary $\in > 0$, there exists a positive integer M such that $M \le n < s$ implies $p(Tx_n, Tx_S) < \in$.

(D) The sequence $\{Tx_n\}$is a cauchy sequence.

3. Main Results

Theorem 3.1

If (X, d)be a metric space, p be a w-distance on $X$and let S and T be $S - JSR(p)$ self mappings of X, satisfying $T(X) \subset S(X)$, (2.2.1), (2.2.2) and for each $z \in X$ with $z \neq Tz$ or $z \neq Sz$

$$\inf\{p(Tx, z) + p(Sx, z) + p(STx, STx) + p(SSx, TTx), x \in X\} \text{ (3.1.1)}$$

Then there is a unique common fixed point of T and S.

Proof

Because$T(X) \subset S(X)$,therefore in X, we can define a sequence $\{x_n\}$such that $Tx_n = Sx_{n+1}$. Since X is complete and$Tx_n = Sx_{n+1}$there exists z in X such that $Tx_n \to z$ and$Sx_n \to z$.

Suppose that $z \neq Tz$or$z \neq Sz$, since $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$, therefore by (A) and the lower semi continuity, we have

$$\lim_{n\to\infty} p(Tx_n, z) = \lim_{n\to\infty} p(Sx_n, z)$$

Now,

$$0 < \inf\{p(Tx, z) + p(Sx, z) + p(TSx, TSx) + p(SSx, TTx), x \in X\}$$
$$\le \inf\{p(Tx_n, z) + p(Sx_n, z) + p(TSx_n, TSx_n) + p(SSx_n, TTx_n)\}$$
$$\le \inf\{p(Tx_n, z) + p(Sx_n, z) + \max[\alpha p(STx_n, TSx_n), \alpha p(SSx_n, TTx_n)] + p(SSx_n, TTx_n)\} < 0.$$

which is a contradiction and hence, our assumption that $z \neq Tz$ or $z \neq Sz$ was wrong. Therefore,$Tz = Sz = z$. Applying (2.2.1)of Lemma 2.2 and (2.3.1), (2.3.2) of Lemma 2.3 uniqueness of the fixed point is obvious.

Theorem 3.2

Let(X, d) be a complete metric space with a w-distance p and let S and T be S-JSR*(p) self mappings of X, satisfying $T(X) \subset S(X)$, (2.2.1) and (2.2.2), for each $z \in X$ with $z \neq Tz$ or $z \neq Sz$

$$\inf\{p(Tx, z) + p(Sx, z) + p(TSx, STx) + p(SSx, TTx), x \in X\} \text{(3.2.1)}$$

Then there is a unique common fixed point of T and S.

Proof

Because $T(X) \subset S(X)$, therefore in X, we can define a sequence $\{x_n\}$such that $Tx_n = Sx_{n+1}$. Since X is complete and$Tx_n = Sx_{n+1}$there exists $z \; in \; X$ such that $Tx_n \to z \; and \; Sx_n \to z$.

Suppose that $z \neq Tz$or $z \neq Sz$, since $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$, therefore by (A) and the lower semi continuity, we have

$$\lim_{n\to\infty} p(Tx_n, z) = \lim_{n\to\infty} p(Sx_n, z)$$

Now,

$$0 < \inf\{p(Tx, z) + p(Sx, z) + p(TSx, TSx) + p(SSx, TTx), x \in X\}$$
$$\le \inf\{p(Tx_n, z) + p(Sx_n, z) + p(TSx_n, TSx_n) + p(SSx_n, TTx_n)\}$$
$$\le \inf\{p(Tx_n, z) + p(Sx_n, z) + \max[\alpha p(STx_n, TSx_n), \alpha p(SSx_n, TTx_n)] + p(SSx_n, TTx_n)\} < 0.$$

which is a contradiction and hence, our assumption that $z \neq Tz$ or $z \neq Sz$ was wrong. Therefore $Tz = Sz = z$. Applying (2.2.1) of Lemma 2.2 and (2.3.1), (2.3.2) of Lemma 2.3 uniqueness of the fixed point is obvious.

4. Examples

Example 4.1

Let $X = [0,1]$ with $d(x, y) = |x - y|$ and S, T are two self mapping on X defined by $Sx = \frac{2}{x+2}, Tx = \frac{1}{x+1}$for $x \in X$. Now we have the sequence $\{x_n\}$ in X is defined as $x_n = \frac{1}{n}$, $n \in N$.Then we have

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 1$$

$|STx_n - Tx_n| \to \frac{1}{3}and|SSx_n - Sx_n| \to \frac{2}{3}$ as $n \to \infty$.

Clearly we have

$|STx_n - Tx_n| < |SSx_n - Sx_n|$.

Thus pair (S, T) is S-JSR mapping. But this pair is neither compatible nor weakly compatible nor other non commuting mapping. Hence pair of JSR mapping is more general than others.

Example 4.2

Let $X = [0,1]$ with $p(x, y) = \max\left\{\left|\frac{x}{2} - y\right|, \frac{1}{2}|x - y|\right\}$ andS, T are two self mapping on X defined by $Sx = \frac{2}{x+2}, Tx = \frac{1}{x+1}$ for $x \in X$.

Now we have the sequence $\{x_n\}$ in X is defined as$x_n = \frac{1}{n}$, $n \in N$. Then we have

$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 1$. Now

$$p(STx_n, Tx_n) = \max\left\{\left|\frac{STx_n}{2} - Tx_n\right|, \frac{1}{2}|STx_n - Tx_n|\right\}$$
$$= \max\left\{\frac{2}{3}, \frac{1}{6}\right\} = \frac{2}{3}$$
$$p(Tx_n, STx_n) = \max\left\{\left|\frac{Tx_n}{2} - Tx_n\right|, \frac{1}{2}|Tx_n - Tx_n|\right\}$$
$$= \max\left\{\frac{1}{6}, \frac{1}{6}\right\} = \frac{1}{6}$$
$$p(SSx_n, Sx_n) = \max\left\{\left|\frac{SSx_n}{2} - Sx_n\right|, \frac{1}{2}|SSx_n - Sx_n|\right\}$$
$$= \max\left\{\frac{2}{3}, \frac{1}{3}\right\} = \frac{2}{3}$$
$$p(Sx_n, SSx_n) = \max\left\{\left|\frac{Sx_n}{2} - SSx_n\right|, \frac{1}{2}|Sx_n - SSx_n|\right\}$$
$$= \max\left\{\frac{1}{6}, \frac{1}{3}\right\} = \frac{1}{3}$$

Clearly pair (S, T) is S-JSR(p) mapping. Also $p(x, y) \neq p(y, x)$.

Example 4.3

Let $X = [0,1]$ with $p(x, y) = \max\{|\frac{x}{2} - y, 12|x-y|\}$ and S, T are two self mapping on X defined by

$Sx = \frac{2}{x+2}, Tx = \frac{1}{x+1}$ for $x \in X$.

Now we have the sequence $\{x_n\}$ in X is defined as $x_n = 1 - \frac{1}{n}$, $n \in N$.Then we have

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 1.$$

In view of Theorem 3.1,$z = 1$ is unique common fixed point of $T$and .

Conclusion

So we have established two fixed point theorems for non-commuting JSR and JSR* mappingsvia w-distance in complete metric space are proved supported with examples.

References

1. *Feng Y. and Liu S: Fixed point theorems for multivalued contractive mappings and multivalued Caristi Type mappings, J. Math. Anal. Appl., 317(2006), pp.103-112.*
2. *Husain T. and Latif A., Fixed points of multivalued non-expansive maps, Internat. J. Math. & Math. Sci., 14(1991), pp.421-430.*
3. *Kada O., Suzuki T., and Takahashi W., Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japan., 44(1996), pp.381-391.*
4. *Kannan R., Some results on fixed points, Bull. math. Calcutta, 6(1968), pp.405-408.*
5. *Kim T.H., Kim K., and Ume J.S., Fixed point theorems on complete metric spaces, Panamer. Math. J., 7(1997), pp.41-51.*
6. *Latif A., and Beg I., Geometric Fixed Points For Single And Multivalued Mappings, Demonstration Mathematica, 30(4) (2006), pp. 791-800.*
7. *Latif A. and Wafaa A. Albar, Fixed points results for Multivalued maps, IJCMS, 23(2) (2007), pp.1129-1136.*
8. *Abdul Latif, Wafaa A. Albar, Fixed Point Results In Complete Metric Spaces, Demonstratio Mathematica Vol. XII No 1(2008).*
9. *Nadler S.B., Multivalued contraction mappings, Pacific J. Math., 30(1969), pp.475-488.*
10. *Naidu S.V.R., (2004), Fixed points and coincidence points for multimaps with not necessarily bounded images, Fixed Point Theory and Appl., 3(2004), pp.221-242.*
11. *Park S., On generalizations of the Ekland-type variational principles, Non-linear Anal., 39(2000),pp.881-889.*
12. *Suzuki T., and Takahashi W., Fixed point Theorems and characterizations of metric completeness, Topol. Methods Nonlinear Anal., 8(1996), 371-382.*
13. *Suzuki, T, Generalized distance and existence theorems in complete metric spaces. J. Math. Anal. Appl. 253(2001), pp.440-458.*
14. *Takahashi W., Existence theorems in metric spaces and characterizations of metric completeness, Josai Math. Monograph, 1(1999), pp.67-85.*
15. *Takahashi W., Nonlinear Functional Analysis: Fixed point theory and its applications, Yokohama Publishers, (2000).*
16. *Ume J.S., Lee B.S., and Cho S.J., Some results on fixed point theorems for multivalued mappings in complete metric spaces, IJMMS, 30(2002), pp.319-325.*